HOMOTOPY TYPE OF MODULI SPACES OF G-HIGGS BUNDLES AND REDUCIBILITY OF THE NILPOTENT CONE

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ABSTRACT. Let G be a real reductive Lie group, and $H^{\mathbb{C}}$ the complexification of its maximal compact subgroup $H \subset G$. We consider classes of semistable G-Higgs bundles over a Riemann surface X of genus $g \geqslant 2$ whose underlying $H^{\mathbb{C}}$-principal bundle is unstable. This allows us to find obstructions to a deformation retract from the moduli space of G-Higgs bundles over X to the moduli space of $H^{\mathbb{C}}$-bundles over X, in contrast with the situation when $g = 1$, and to show reducibility of the nilpotent cone of the moduli space of G-Higgs bundles, for G complex.

1. INTRODUCTION

A *Higgs bundle* on a Riemann surface X is a pair (E, φ), where E is a rank n holomorphic vector bundle over X and $\varphi \in H^0(\mathrm{End}(E) \otimes K)$ is a holomorphic endomorphism of E twisted by the canonical bundle K of X. Higgs bundles appeared first in the work of Hitchin [Hi87] and Simpson [Si92, Si88]. The non-abelian Hodge Theorem [Co88, Do87, Hi87, Si88] identifies the moduli space of Higgs bundles with the *character variety* for representations of the fundamental group of X into $\mathrm{GL}(n, \mathbb{C})$.

The appropriate objects for extending the non-abelian Hodge Theorem to representations of the fundamental group in a *real* reductive Lie group G (see, e.g., [Hi92, GGM09, Go14]) are called *G-Higgs bundles*. There are natural notions of stability, semistability, and polystability for G-Higgs bundles, leading to corresponding moduli spaces $\mathcal{M}(G)$ (see [GGM09] for the general theory). Again, there is an identification between $\mathcal{M}(G)$ and the moduli space of flat G-connections on X.

Motivated partially by this identification, the moduli space of G-Higgs bundles has been extensively studied. When G is a complex semisimple Lie group Biswas and Florentino proved in [BF11] that the moduli space of topologically trivial principal G-bundles over a compact Riemann surface (which are actually H-Higgs bundles, where H is the maximal compact subgroup of G) is not a deformation retraction of the moduli space of topologically trivial G-Higgs bundles. This result contrasts with the main theorem of Florentino and Lawton [FL09] which says that the moduli space of flat H-connections on an *open surface* X is a strong deformation retraction of the moduli space of flat G-connections on X, for complex reductive G.

Our aim in this paper is to generalize the above mentioned theorem of Biswas and Florentino to the case of real reductive Lie groups. Using the non-abelian Hodge theorem, the question is to prove that the moduli spaces of semistable principal $H^{\mathbb{C}}$-bundles, which we denote by $\mathcal{N}(H^{\mathbb{C}})$, is not a deformation retraction of the moduli spaces of semistable G-Higgs bundles $\mathcal{M}(G)$, where $H^{\mathbb{C}}$ is the complexification of H. We recall that the

1991 *Mathematics Subject Classification.* 14H60, 32L05.

Key words and phrases. Higgs bundles, Character varieties, Nilpotent cone, Topology of moduli spaces.

This work was partially supported by CMUP (UID/MAT/00144/2013), CMAF-CIO (MAT/04561), the projects PTDC/MAT-GEO/2823/2014 and EXCL/MAT-GEO/0222/2012 funded by FCT (Portugal) with national funds and a grant from IPM. The authors acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network).

topological invariants of the underlying principal bundles label unions of connected components of the moduli spaces, so in order to study deformation retraction from $\mathcal{M}(G)$ to $\mathcal{N}(H^{\mathbb{C}})$ we should consider separately each topological type. In this paper, we address the case of trivial topological type.

Our strategy is as follows. We use the $\mathbb{C}^*$-action on the moduli space of G-Higgs bundles, given by multiplication of the Higgs field, and show (Proposition 2.10) that it provides a deformation retraction onto the *nilpotent cone*: the pre-image of zero under the Hitchin map, defined in section 2.4.1. Therefore, we reduce the question to finding obstructions to a deformation from the nilpotent cone to $\mathcal{N}(H^{\mathbb{C}})$. Then we prove that such obstructions are semistable G-Higgs bundles whose underlying $H^{\mathbb{C}}$-bundle is unstable and we show existence of these obstructions by using the construction of [GPR15], stated in Proposition 3.9. This result allows us also to deduce the reducibility of the nilpotent cone of the moduli space of G-Higgs bundles when G is a connected reductive complex Lie group.

More precisely, our main results are the following theorems (see Theorems 3.12 and 3.15 below; note that the moduli spaces may be singular).

Theorem A. *Let G be a non-abelian connected reductive complex Lie group. Then the nilpotent cone in the moduli space of G-Higgs bundles of trivial topological type is not irreducible.*

Theorem B. *Let G be a non-abelian (real or complex) connected reductive Lie group of non-Hermitian type or connected simple real Lie group of Hermitian non-tube type. Then the moduli space of semistable principal $H^{\mathbb{C}}$-bundles of trivial topological type is not a deformation retraction of the moduli space of semistable G-Higgs bundles of trivial topological type.*

2. Moduli of Higgs bundles and the nilpotent cone

2.1. **G-Higgs bundles.** Let X be a compact connected Riemann surface of genus g, for $g \geqslant 2$, and let $K = T^*X$ be the canonical bundle of X. Let G be a (real or complex) connected reductive Lie group with a choice of a maximal compact subgroup $H \subset G$, and denote by $H^{\mathbb{C}}$ the complexification of H.

By an *$H^{\mathbb{C}}$-bundle over X* we always mean a *holomorphic* principal $H^{\mathbb{C}}$-bundle over X. Recall that this is a holomorphic fibre bundle $\pi : E \to X$ with a holomorphic $H^{\mathbb{C}}$-action which is free and transitive on each fibre and E is required to admit holomorphic $H^{\mathbb{C}}$-equivariant local trivializations $E|_U \cong U \times H^{\mathbb{C}}$ over small open sets $U \subset X$. Denote by $\mathcal{N}(H^{\mathbb{C}})$ the moduli space of semistable principal $H^{\mathbb{C}}$-bundles over X; the construction of the moduli space can be found in [Ra96]. It is a union of connected components (see [Ra75])

$$\mathcal{N}(H^{\mathbb{C}}) = \coprod_d \mathcal{N}_d(H^{\mathbb{C}})$$

indexed by the elements $d \in \pi_1(H^{\mathbb{C}})$ which correspond to topological types of principal $H^{\mathbb{C}}$-bundles E over X. Moreover, for each $d \in \pi_1(H^{\mathbb{C}})$, $\mathcal{N}_d(H^{\mathbb{C}})$ is non-empty.

If $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ are the corresponding Lie algebras, there is a (complexified) Cartan decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

where $\mathfrak{m}^{\mathbb{C}}$ is a complex vector space. The restriction of the adjoint representation $\mathrm{Ad} : G^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{g}^{\mathbb{C}})$ to $H^{\mathbb{C}}$ preserves the Cartan decomposition and induces the *isotropy representation* of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}}$:

$$\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}}). \tag{2.1}$$

Given a $H^{\mathbb{C}}$-bundle E, denote by $E(\mathfrak{m}^{\mathbb{C}})$ the vector bundle with fibres $\mathfrak{m}^{\mathbb{C}}$ associated to E via the isotropy representation, i.e., $E(\mathfrak{m}^{\mathbb{C}}) = E \times_\iota \mathfrak{m}^{\mathbb{C}}$.

Definition 2.1. A *G-Higgs bundle* on a Riemann surface X is a pair (E, φ) which consists of a principal $H^{\mathbb{C}}$-bundle E and a holomorphic section φ of the bundle $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$. The section φ is called the *Higgs field*.

Remark. We have the following particular cases.

(1) If G is itself a compact group, then $\mathfrak{m}^{\mathbb{C}} = 0$, so the Higgs field is identically zero, and we recover the notion of principal $H^{\mathbb{C}} = G^{\mathbb{C}}$-bundle.
(2) When G is a complex group, then we have $H^{\mathbb{C}} = G$ and also $\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}$. So, a *$G$-Higgs bundle* is a pair (E, φ), where E is a G-bundle and $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K) = H^0(X, \mathrm{Ad}(E) \otimes K)$.
(3) When G is non-compact of Hermitian type there is an almost complex structure on $\mathfrak{m}^{\mathbb{C}}$ defined by the adjoint action of a special element J in the center $\mathfrak{z}$ of $\mathfrak{h}$ with $J^2 = -\mathrm{id}$. The almost complex structure splits $\mathfrak{m}^{\mathbb{C}}$ into $H^{\mathbb{C}}$-invariant $\pm i$-eigenspaces
$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$$
and therefore splits the bundle $E(\mathfrak{m}^{\mathbb{C}}) = E(\mathfrak{m}^+) \oplus E(\mathfrak{m}^-)$. Hence the Higgs field decomposes as $\varphi = (\varphi^+, \varphi^-)$ where

$$\varphi^+ \in H^0(X, E(\mathfrak{m}^+) \otimes K), \quad \varphi^- \in H^0(X, E(\mathfrak{m}^-) \otimes K). \tag{2.2}$$

The notion of G-Higgs bundle includes several interesting particular cases. When G is a classical Lie group, G-Higgs bundles can be defined in terms of holomorphic vector bundles with additional structure, as follows.

Example 2.2. A $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle on X is a pair (E, φ), where E is a rank n holomorphic vector bundle over X and $\varphi \in H^0(\mathrm{End}(E) \otimes K)$ is a holomorphic endomorphism of E twisted by K. This is the original notion of Higgs bundle introduced by Hitchin [Hi87]. Similarly, a $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle is a pair (E, φ), where $E \to X$ is a holomorphic rank n vector bundle with $\det(E) = \mathcal{O}$ and $\varphi \in H^0(X, \mathrm{End}(E) \otimes K)$ with $\mathrm{tr}(\varphi) = 0$.

Example 2.3. A $\mathrm{SO}(n, \mathbb{C})$-Higgs bundle is a pair (E, φ) where E is a $\mathrm{SO}(n, \mathbb{C})$-bundle and $\varphi \in H^0(E(\mathfrak{so}(n, \mathbb{C})) \otimes K)$. Using the standard representations of $\mathrm{SO}(n, \mathbb{C})$ in $\mathbb{C}^n$ we can associate to E a holomorphic vector bundle W of rank n with trivial determinant,
$$W = E \times_{\mathrm{SO}(n,\mathbb{C})} \mathbb{C}^n,$$
together with a non-degenerate symmetric quadratic form $Q \in H^0(\mathrm{S}^2 W^*)$; we can think of Q as a symmetric holomorphic isomorphism $Q : W \to W^*$. The Higgs field in terms of the vector bundle W is a holomorphic section $\varphi \in H^0(\mathrm{End}(W) \otimes K)$ satisfying $Q(u, \varphi v) = -Q(\varphi u, v)$ and $\mathrm{tr}(\varphi) = 0$.

Example 2.4. Let $G = \mathrm{SL}(n, \mathbb{R})$. The Cartan decomposition of the Lie algebra is given by
$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{m},$$
where $\mathfrak{m} = \{\text{symmetric real matrices of trace } 0\}$. So a $\mathrm{SL}(n, \mathbb{R})$-Higgs bundle is a pair (E, φ), where E is a $\mathrm{SO}(n, \mathbb{C})$-bundle and $\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$. Hence a $\mathrm{SL}(n, \mathbb{R})$-Higgs bundle can be viewed as a triple (W, Q, φ), where (W, Q) is a holomorphic orthogonal bundle with $\det(W) = \mathcal{O}$, and φ is a traceless holomorphic section of $\mathrm{End}(W) \otimes K$ that is symmetric with respect to Q, i.e. $Q\varphi^T Q = \varphi$.

2.2. **Moduli spaces.** For the construction of moduli spaces, as usual one introduces several notions of stability. The notions of stability, semistability and polystability for G-Higgs bundles depend on a real parameter α and generalize the usual slope stability condition for Higgs bundles and Ramanathan's stability condition for principal bundles. In the present work we consider only the particular case $\alpha = 0$, because this is the relevant value for relating G-Higgs bundles to representations of $\pi_1(X)$ via the non-abelian Hodge theorem. Thus we simply say polystable instead of 0-polystable and likewise for stable and semistable, and refer the reader to [GGM09] for the general definitions.

Remark 2.5. To a G-Higgs bundle, for $G \subset \mathrm{GL}(n, \mathbb{C})$, we can naturally associate a $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle. By this correspondence, semistability of a G-Higgs bundle is equivalent to semistability of the associated $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle. For stability the situation is more subtle: it is possible for a stable G-Higgs bundle to induce a strictly semistable $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle.

2.3. **Components of moduli spaces.** To a given G-Higgs bundle we can associate the topological invariant of the underlying $H^{\mathbb{C}}$-bundle. As mentioned before, for connected $H^{\mathbb{C}}$, topological types are well-known [Ra75] to be classified by elements of

$$\pi_1(H^{\mathbb{C}}) = \pi_1(H) = \pi_1(G).$$

Definition 2.6. For a fixed $d \in \pi_1(G)$, *the moduli space of polystable G-Higgs bundles* $\mathcal{M}_d(G)$ is defined to be the set of isomorphism classes of polystable G-Higgs bundles (E, φ) with $c(E) = d$.

These moduli spaces are complex algebraic varieties, due to constructions of Schmitt [Sc05, Sc08]. We have the disjoint union

$$\mathcal{M}(G) = \coprod_d \mathcal{M}_d(G).$$

When G is a complex reductive Lie group the moduli space $\mathcal{M}_d(G)$ is connected and non-empty, for every $d \in \pi_1(G)$ (see [GO16]). But the situation is very different when G is a real reductive Lie group. In this case the moduli space $\mathcal{M}_d(G)$ can be a union of several connected components and can also be empty for some $d \in \pi_1(G)$.

The following are three known cases of real Lie groups for which there exists a topological type d such that $\mathcal{M}_d(G)$ is disconnected:

(1) When G is a split real form, proved by Hitchin [Hi92],
(2) When G is non-compact of Hermitian type, the *Cayley correspondence* [BGG06] provides extra components in the moduli space for maximal Toledo invariant (defined below). For $G = \mathrm{SL}(2, \mathbb{R})$, this goes back to Goldman [Gol80].
(3) When $G = \mathrm{SO}_0(p, q)$ there are, in general, extra components not accounted for by the preceding mechanisms, see [Co17, ABCGGO18].

In the case when G is non-compact of Hermitian type one can define an integer invariant $\tau(E, \varphi)$ called the *Toledo invariant* which is an element of the torsion free part of $\pi_1(H)$. This invariant is bounded by a *Milnor-Wood inequality*, beyond which the moduli spaces are empty. In fact, if G is non-compact of Hermitian type and $(E, \varphi^+, \varphi^-)$ is a semistable G-Higgs bundle, then the Toledo invariant $\tau = \tau(E)$ satisfies

$$-\operatorname{rk}(\operatorname{im}(\varphi^+))(2g-2) \leqslant \tau \leqslant \operatorname{rk}(\operatorname{im}(\varphi^-))(2g-2), \tag{2.3}$$

(see [BGR, GN]) where φ^+, φ^- are defined in (2.2).

Example 2.7. A $\mathrm{SL}(2, \mathbb{R})$-Higgs bundle has the form

$$E = (W = L \oplus L^*, Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \varphi = \begin{pmatrix} 0 & \varphi^+ \\ \varphi^- & 0 \end{pmatrix}),$$

where L is a line bundle, $\varphi^+ \in H^0(X, L^2 \otimes K)$ and $\varphi^- \in H^0(X, L^{-2} \otimes K)$. The group $\mathrm{SL}(2,\mathbb{R})$ is of Hermitian type and the Toledo invariant is $\tau(E) = 2\deg(L)$. The inequality (2.3) implies $|\deg(L)| \leqslant g-1$ and, if both φ^+ and φ^- are non-zero, any E satisfying this inequality is semistable. Moreover, if $\varphi^+ = 0$, then E is semistable if and only if $\deg(L) \geqslant 0$ and if $\varphi^- = 0$, then E is semistable if and only if $\deg(L) \leqslant 0$. Thus, if the Higgs field vanishes then E is semistable if and only if $\deg(L) = 0$.

Remark 2.8. It follows from (2.3) that, if G is of Hermitian type and $(E, 0)$ is a semistable G-Higgs bundle, then $\tau(E) = 0$.

For all the real connected semisimple classical groups of Hermitian type, namely $\mathrm{SU}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(2n)$ and $\mathrm{SO}_0(2,n)$ we have $\pi_1(H) \cong \mathbb{Z}$, except $G = \mathrm{SO}_0(2,n)$ with $n \geqslant 3$ for which $\pi_1(H) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. So in these cases, i.e. excepting $\mathrm{SO}_0(2,n)$, the topological type of the G-Higgs bundle is determined by the Toledo invariant.

2.4. The $\mathbb{C}^*$-action on the moduli spaces and retraction to the nilpotent cone.

In this subsection we show how the use of a $\mathbb{C}^*$-action on the moduli space of G-Higgs bundles implies a deformation retraction onto the nilpotent cone.

The moduli space of G-Higgs bundles $\mathcal{M}_d(G)$ admits a non-trivial holomorphic $\mathbb{C}^*$-action [Hi87, Si92] by multiplication of the Higgs field,

$$z \cdot (E, \varphi) = (E,\ z\varphi). \tag{2.4}$$

From the gauge theory point of view one can observe that the action of the subgroup $S^1 \subset \mathbb{C}^*$ on the moduli space is Hamiltonian with proper moment map defined as follows

$$\begin{aligned} f : \mathcal{M}_d(G) &\to \mathbb{R} \\ (E, \varphi) &\mapsto \|\varphi\|^2 := \int_X |\varphi|^2 \mathrm{vol}. \end{aligned}$$

When the moduli space $\mathcal{M}_d(G)$ is smooth, the theorem of Frankel [Fr59] implies that f is a perfect Bott-Morse function. Another consequence of the fact that f is a moment map for the Hamiltonian S^1-action is that the set of critical points of f coincides with the set of fixed points of the action. We also recall that the sets of fixed points of the actions of S^1 and $\mathbb{C}^*$ coincide. Let $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ be the set of the irreducible components of the fixed point set of the $\mathbb{C}^*$-action on $\mathcal{M}_d(G)$, with Λ an index set.

There exists a *Morse stratification* on the moduli spaces $\mathcal{M}_d(G)$ which coincides with ther *Białynicki-Birula stratification*, due to results of Kirwan in [Ki84]. It is defined as follows. Let

$$U_\lambda := \{(E, \varphi) \in \mathcal{M}_d(G) \mid \lim_{z \to 0} z \cdot (E, \varphi) \in \mathcal{F}_\lambda\}.$$

Then $\cup_\lambda U_\lambda$ gives a stratification of $\mathcal{M}_d(G)$.

One can also define the so-called *downward Morse flow* of $\mathcal{F}_\lambda$ which, again due to the result of Kirwan, is given by the sets $D_\lambda := \{(E, \varphi) \in \mathcal{M}_d(G) \mid \lim_{z \to \infty} z \cdot (E, \varphi) \in \mathcal{F}_\lambda\}$. Using the label $0 \in \Lambda$ to denote the fixed point set of G-Higgs bundles with zero Higgs field, it is clear that we have $\mathcal{F}_0 = \mathcal{N}_d(H^{\mathbb{C}})$. Note that $\mathcal{M}_d(G)$ does not have to be smooth for the Białynicki-Birula stratification to be defined.

2.4.1. *Nilpotent Cone.* Take a basis $\{\beta_1, \cdots, \beta_r\}$ for the G-invariant polynomials on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ (under the adjoint action) and let $d_i = \deg(\beta_i)$. Given a G-Higgs bundle (E, φ), the evaluation of β_i on φ gives a section $\beta_i(\varphi) \in H^0(X, K^{d_i})$. For a fixed $d \in \pi_1(G)$

the (restricted) *Hitchin map* is defined to be

$$\begin{aligned} \mathcal{H} : \mathcal{M}_d(G) &\to \bigoplus H^0(X, K^{d_i}) \\ (E,\varphi) &\mapsto (\beta_1(\varphi), \dots, \beta_r(\varphi)). \end{aligned}$$

For example when $G = \mathrm{GL}(n,\mathbb{C})$ then $\beta_i(\varphi)$ can be taken to be $\mathrm{tr}(\wedge^i \varphi)$ and $d_i = i$ for all $i = 1, \dots, n$. The Hitchin map is proper for any choice of basis; see [Hi87, Hi92]. A more general direct construction (i.e. without passing to the complex group) of the Hitchin map for real G can be found in [GPR15].

The pre-image of zero under the Hitchin map $\mathcal{H}^{-1}(0) \subset \mathcal{M}_d(G)$ is called the *nilpotent cone*. This was defined by Laumon [La88] in the case of a complex group, and by abuse of language we use the same name when G is a real Lie group. The Hitchin map is algebraic, so the nilpotent cone is a subscheme which is, in general, neither reduced nor irreducible (see [Hi17] for a precise analysis in the case $G = \mathrm{SL}(2,\mathbb{C})$). However, we shall view it as a subvariety[1], i.e., we consider the associated reduced scheme.

Proposition 2.9. [Ha] *The downward Morse flow coincides with the nilpotent cone, more precisely*

$$\mathcal{H}^{-1}(0) = \bigcup_{\lambda \in \Lambda} \bar{D}_\lambda.$$

From the above proposition and the fact that $\mathcal{H}$ is proper we can also deduce that each component of the nilpotent cone is a projective variety. The following result generalizes the one for semisimple complex G given in [BF11], with an analogous proof.

Proposition 2.10. *Let G be a real reductive Lie group. Then the nilpotent cone $\mathcal{H}^{-1}(0)$ is a deformation retraction of the moduli space $\mathcal{M}_d(G)$.*

Proof. Fixing a Hermitian metric on X it induces a Hermitian metric on K, and hence a inner product on each vector space $H^0(X, K^{d_i})$. Consider the following composition map:

$$\begin{aligned} \mathcal{M}_d(G) \xrightarrow{\mathcal{H}} \bigoplus_{i=1}^{r} H^0(X, K^{d_i}) &\xrightarrow{f} \mathbb{R}_{\geqslant 0}, \\ (s_1, \cdots, s_r) &\mapsto \sum_{i=1}^{r} \|s_i\|^{\frac{1}{d_i}}. \end{aligned}$$

Since both the Hitchin map $\mathcal{H}$ and f are proper, the inverse image $(f \circ \mathcal{H})^{-1}([0,\epsilon]) =: U_\epsilon$ is a compact neighborhood of the nilpotent cone. Note that for any real $t \geqslant 0$ and $s_i \in H^0(X, K^{d_i})$ we have $\|t.s_i\| = t^{d_i}\|s_i\|$ and hence

$$f(ts_1, \cdots, ts_r) = t\, f(s_1, \cdots, s_r) \tag{2.5}$$

Using the $\mathbb{C}^*$-action on the moduli space of G-Higgs bundles (2.4) we define the following homotopy between the identity map of $\mathcal{M}_d(G)$ and a retraction onto U_ϵ as follows:

$$\begin{aligned} \mathcal{F} : \mathcal{M}_d(G) \times [0,1] &\to \mathcal{M}_d(G) \\ (E,\varphi) &\mapsto \begin{cases} \begin{cases} (E, t_0 \cdot \varphi) & t \leqslant t_0 := \frac{\epsilon}{f(\mathcal{H}(E,\varphi))} \\ (E, t \cdot \varphi) & t > t_0 \end{cases} & \text{if } f(\mathcal{H}(E,\varphi)) > \epsilon \\ (E,\varphi) & \text{if } f(\mathcal{H}(E,\varphi)) \leqslant \epsilon \end{cases} \end{aligned}$$

[1]We shall not require varieties to be irreducible.

Indeed, we have

$$\mathcal{F}((E,\varphi),t) = (E,\varphi), \text{ for } (E,\varphi) \in U_\epsilon$$
$$\mathcal{F}((E,\varphi),1) = (E,\varphi), \text{ for } (E,\varphi) \in \mathcal{M}_d(G).$$

Next we prove $\mathcal{F}((E,\varphi),0) \in U_\epsilon$ to conclude that U_ϵ is a deformation retraction of $\mathcal{M}_d(G)$. Clearly if $f(\mathcal{H}(E,\varphi)) \leqslant \epsilon$ then $\mathcal{F}((E,\varphi),0) = (E,\varphi) \in U_\epsilon$. If $f(\mathcal{H}(E,\varphi)) > \epsilon$ then

$$f(\mathcal{H}(\mathcal{F}((E,\varphi),0))) = f(\mathcal{H}(E,t_0\cdot\varphi)) = t_0\, f(\mathcal{H}(E,\varphi)) = \epsilon,$$

in the last equality we use the equality (2.5).

The nilpotent cone is a proper subvariety of $\mathcal{M}_d(G)$ so it is a finite CW-complex and an absolute deformation retract (see, for example, [BCR98]). Hence, there is some open neighborhood $U \supseteq \mathcal{H}^{-1}(0)$ such that U deformation retracts to $\mathcal{H}^{-1}(0)$. Choose ϵ small enough so that $U_\epsilon \subset U$, this is possible as $\mathcal{H}$ is proper. Therefore the composition of deformation retraction of U into the nilpotent cone and of $\mathcal{M}_d(G)$ into U_ϵ gives a retraction of $\mathcal{M}_d(G)$ into the nilpotent cone. □

3. The obstructions to a deformation retraction

For every topological type $d \in \pi_1(H)$ there is a natural inclusion $\mathcal{N}_d(H^{\mathbb{C}}) \subset \mathcal{M}_d(G)$ which comes from considering principal $H^{\mathbb{C}}$-bundles as G-Higgs bundles with zero Higgs field. Thus, we have

$$\mathcal{N}_d(H^{\mathbb{C}}) \subset \mathcal{H}^{-1}(0) \subset \mathcal{M}_d(G),$$

and we can identify $\mathcal{N}_d(H^{\mathbb{C}}) = \mathcal{F}_0$. Thus, in order to discuss obstructions to the deformation retraction from the moduli spaces of G-Higgs bundles to $\mathcal{N}_d(H^{\mathbb{C}})$, by using Proposition 2.10, it is enough to study the obstructions to deformation retraction from the nilpotent cone to $\mathcal{N}_d(H^{\mathbb{C}}) = \mathcal{F}_0$, which we do next.

Remark. In the case when G is non-compact of Hermitian type, by Remark 2.8 the right question to ask would be the deformation retraction from $\mathcal{M}_d(G)$ to $\mathcal{N}_d(H^{\mathbb{C}})$ for trivial topological type $d = 0$.

3.1. **Additive homology of $\mathcal{M}_d(G)$.** In this section we consider homology with $\mathbb{C}$-coefficients. The following lemmas are of course well known but, for completeness, we include proofs.

Recall that we do not require algebraic varieties to be irreducible. We understand the dimension of a variety Y to be the maximal dimension of an irreducible component of Y. We also recall that any projective variety has the structure of a finite CW-complex and that this can taken to be compatible with any given subvariety [BCR98, Hir75]. Finally we recall that any irreducible projective variety Y of dimension r has a non-zero fundamental class $[Y] \in H_{2r}(Y) \cong \mathbb{C}$ (see, e.g., [Ha75, II.7.6]) and that $H_n(Y) = 0$ for $n > 2\dim(Y)$.

Lemma 3.1. *Let Y be a projective variety of dimension r. Then $H_{2r}(Y) \cong \mathbb{C}^n$, where n is the number of irreducible components of Y of dimension r.*

Proof. We prove the result by induction on the number of irreducible components of Y, the case $n = 1$ being the result described in the paragraph preceding the lemma. So let $Y = Y_1 \cup Y_2$, where Y_1 is irreducible and Y_2 has $n-1$ irreducible components.

Let $r = \dim(Y)$. Since $\dim(Y_1 \cap Y_2) < r$ we have $H_n(Y_1 \cap Y_2) = 0$ for $n > 2r-2$. Thus the Mayer–Vietoris sequence for $Y = Y_1 \cup Y_2$ gives

$$0 \to H_{2r}(Y_1) \oplus H_{2r}(Y_2) \xrightarrow{\cong} H_{2r}(Y) \to 0.$$

Since by induction the desired result holds for Y_1 and Y_2, the lemma follows. □

Lemma 3.2. *Let Y be a projective variety and suppose that $Y = Y_1 \cup Y_2$, where $Y_i \subsetneq Y$ is non-empty and closed for $i = 1, 2$. Then Y_i and Y have non-isomorphic homology for $i = 1, 2$.*

Proof. Let $r = \dim(Y)$. If both Y_1 and Y_2 have dimension r, the result is immediate from Lemma 3.1. It remains to consider the case when $\dim(Y_1) = s < r$ and $\dim(Y_2) = r$, say. Since clearly Y and Y_1 have distinct homology we just have to show that Y and Y_2 have distinct homology. For this, note first that we may remove any irreducible components of Y_1 which are contained in Y_2 and still have the hypotheses of the Lemma satisfied. Then, by decomposing into irreducible components, we see that $\dim(Y_1 \cap Y_2) < s$. Therefore we have $H_n(Y_1 \cap Y_2) = 0$ for $n > 2s - 2$. Thus the Mayer–Vietoris sequence for $Y = Y_1 \cup Y_2$ gives

$$0 \to H_{2s}(Y_1) \oplus H_{2s}(Y_2) \xrightarrow{\cong} H_{2s}(Y) \to 0.$$

Since, by Lemma 3.1, $H_{2s}(Y_1) \neq 0$, we see that $H_{2s}(Y_2)$ and $H_{2s}(Y)$ are distinct, as desired. □

Lemma 3.3. *Assume that there exists a component $\mathcal{F}_\lambda$ of the fixed locus with $\lambda \neq 0$. Then we may choose $\lambda \neq 0$ such that*

$$\bar{D}_\lambda \cap \mathcal{F}_0 = \left\{ \lim_{z \to 0} (E, z\varphi) \;\mid\; (E, \varphi) \in D_\lambda \right\}.$$

Proof. Following Simpson [Si94, S11], we may consider a $\mathbb{C}^*$-equivariant embedding of $\mathcal{H}^{-1}(0)$ as a projective variety, where the ambient projective space has a standard positively weighted $\mathbb{C}^*$-action and $\mathcal{F}_0$ lies in the weight zero subspace. Then the component $\mathcal{F}_\lambda$ with the lowest non-zero weight of the $\mathbb{C}^*$-action satisfies the condition of the lemma. □

Proposition 3.4. *Suppose that there is a non-empty $\mathcal{F}_\lambda$, for some $\lambda \neq 0$. Then $\mathcal{M}_d(G)$ and $\mathcal{N}_d(H^{\mathbb{C}})$ have distinct additive singular homology.*

Proof. Consider the closed subspace $\bar{D}_\lambda \subset \mathcal{H}^{-1}(0)$. If $\mathcal{F}_0$ is not contained in $\bar{D}_\lambda$ then Lemma 3.2 gives the conclusion. Otherwise Lemma 3.3 tells us that, for suitable λ, any $(E, 0) \in \mathcal{F}_0$ is of the form $(E, 0) = \lim_{z \to 0}(E, z\varphi)$ with $(E, \varphi) \in D_\lambda$. Now consider the $\mathbb{C}^*$-invariant subspace of $\bar{D}_\lambda$,

$$\bar{D}^0_\lambda = \{(E, \varphi) \in \bar{D}_\lambda \;\mid\; \lim_{z \to 0}(E, z\varphi) \in \mathcal{F}_0\}.$$

By Lemma 3.3, the map $\bar{D}^0_\lambda \to \mathcal{F}_0$ given by $(E, \varphi) \mapsto (E, 0)$ is a surjective morphism. Moreover, it is clearly $\mathbb{C}^*$-equivariant. Hence, since the $\mathbb{C}^*$-action is non-trivial on $\bar{D}^0_\lambda$ and trivial on $\mathcal{F}_0$, we conclude that $\dim \bar{D}_\lambda > \dim \mathcal{F}_0$. Therefore Lemma 3.1 shows that $\mathcal{H}^{-1}(0)$ and $\mathcal{F}_0$ have distinct homology, as was to be shown. □

Corollary 3.5. *Suppose that there exists a semistable G-Higgs bundle (E, φ) for which E is unstable as a principal $H^{\mathbb{C}}$-bundle. Then $\mathcal{M}_d(G)$ and $\mathcal{N}_d(H^{\mathbb{C}})$ have distinct additive singular homology.*

Proof. Let (E, φ) be a semistable Higgs bundle and suppose $\lim_{t \to 0}(E,\, t\varphi)$ is $(E,\, 0)$. Then E is a semistable $H^{\mathbb{C}}$-bundle. So, our hypothesis implies $\lim_{t \to 0}(E,\, t\varphi) = (E_0, \varphi_0)$ with $\varphi_0 \neq 0$. Therefore $(E_0, \varphi_0) \in \mathcal{F}_\lambda$, with $\lambda \neq 0$ (as $||\varphi_0|| \neq 0$) and hence the result follows using Proposition 3.4. □

3.2. **The associated Higgs bundle.** The following can be found in [Hi92, GPR15]. Again, let G be a real reductive Lie group with maximal compact subgroup H, and $\mathfrak{g}^{\mathbb{C}}$ be the complexification of the Lie algebra $\mathfrak{g}$ of G. Let $\sigma\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be the corresponding ($\mathbb{C}$-antilinear) real structure and let $\theta\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be the ($\mathbb{C}$-linear) Cartan involution. Consider the Cartan decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

into ± 1-eqigenspace for θ.

For example, for $G = \mathrm{SL}(2,\mathbb{R})$, the Cartan involution is $\theta\colon X \mapsto -X^t$ and the Cartan decomposition of $\mathfrak{sl}(2,\mathbb{C})$ under θ is

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{so}(2,\mathbb{C}) \oplus \mathfrak{sym}_0(2,\mathbb{C})$$

where $\mathfrak{so}(2,\mathbb{C})$ denotes the trace zero complex diagonal matrices, and $\mathfrak{sym}_0(2,\mathbb{C})$ the complex antidiagonal matrices.

When G is non-abelian, there is a σ and θ-equivariant injective morphism

$$\rho' : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}^{\mathbb{C}},$$

such that $\rho' = \rho'_+ \oplus \rho'_-$, where

$$\rho'_+ : \mathfrak{so}(2,\mathbb{C}) \to \mathfrak{h}^{\mathbb{C}}, \quad \rho'_- : \mathfrak{sym}_0(2,\mathbb{C}) \to \mathfrak{m}^{\mathbb{C}}. \tag{3.1}$$

Since $\mathrm{SL}(2,\mathbb{C})$ is simply-connected ρ' lifts to

$$\rho : \mathrm{SL}(2,\mathbb{C}) \to G^{\mathbb{C}}. \tag{3.2}$$

On the other hand, the restriction $\rho'|_{\mathfrak{sl}(2,\mathbb{R})} : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ lifts to a θ-equivariant group homomorphism, still denoted by ρ

$$\rho : \mathrm{SL}(2,\mathbb{R}) \to G \tag{3.3}$$

which takes $\mathrm{SO}(2)$ to H. We denote by ρ_+ the complexification of the restriction $\rho|_{\mathrm{SO}(2)}$

$$\rho_+ : \mathrm{SO}(2,\mathbb{C}) \to H^{\mathbb{C}}. \tag{3.4}$$

given an $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle (E',φ') we can construct a G-Higgs bundle (E,φ) via (3.4) and (3.1) in the following way:

$$E := E' \times_{\mathrm{SO}(2,\mathbb{C})} H^{\mathbb{C}}, \quad \varphi := \rho'_-(\varphi') \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K). \tag{3.5}$$

More generally we have the following: let $f : G' \to G$ be a morphism of reductive Lie groups. This induces a morphism $f : H'^{\mathbb{C}} \to H^{\mathbb{C}}$, still denoted by the same symbol. Given a G'-Higgs bundle (E',φ') one can associate a G-Higgs bundle (E,φ), which is called the *extended G-Higgs bundle* via f, with $E := E' \times_{H'^{\mathbb{C}}} H^{\mathbb{C}}$. Moreover, since $\varphi' \in H^0(X, E(\mathfrak{m}'^{\mathbb{C}}) \otimes K)$ we get a section of

$$E(\mathfrak{m}^{\mathbb{C}}) := E(\mathfrak{m}'^{\mathbb{C}}) \underset{i\circ F}{\times} \mathfrak{m}^{\mathbb{C}},$$

where i is the isotropy representation for G'. Hence φ' defines via the homomorphism $F := D_e f : \mathfrak{g}' \to \mathfrak{g}$, a Higgs field φ on E.

When G is connected, f induces a homomorphism between the fundamental groups

$$f_* : \pi_1(G') \to \pi_1(G)$$

and the topological type of the associated G-Higgs bundle corresponds to the image via the map f_*. We recall the following result on polystability for the associated G-Higgs bundle:

Proposition 3.6. *Let $f : G' \to G$ be a morphism between reductive Lie groups (real or complex). Let (E', φ') be a G'-Higgs bundle and (E, φ) be the extended G-Higgs bundle via f. Then, if (E', φ') is semistable, then so is (E, φ). Thus the group homomorphism f defines a morphism*

$$\begin{array}{rcl} \mathcal{M}_d(G') & \to & \mathcal{M}_{f_* d}(G) \\ (E', \varphi') & \mapsto & (E, \varphi) \end{array}$$

Proof. This follows from [GPR15, Corollary 5.10], since the stability parameter is zero. □

Proposition 3.7. *Let $H'^{\mathbb{C}}$ and $H^{\mathbb{C}}$ be connected complex Lie groups with $H'^{\mathbb{C}}$ semisimple and $H^{\mathbb{C}}$ reductive. Let $f : H'^{\mathbb{C}} \to H^{\mathbb{C}}$ be a morphism with discrete kernel. Let E' be a principal $H'^{\mathbb{C}}$-bundle and let E be the principal $H^{\mathbb{C}}$-bundle obtained by extension of structure group via f. If E' is unstable as a $H'^{\mathbb{C}}$-bundle, then E is unstable as a $H^{\mathbb{C}}$-bundle.*

Proof. Since $H'^{\mathbb{C}}$ is semisimple, the unstable $H'^{\mathbb{C}}$-bundle E' is destabilized by a reduction to a proper parabolic subgroup[2]. Now, if f is surjective, the result follows from [Ra75, Proposition 7.1] — note that one needs to ensure that the image of a proper parabolic in $H'^{\mathbb{C}}$ is a proper parabolic in $H^{\mathbb{C}}$, and the hypothesis on the kernel of f achieves this. For the general case, suppose then that E' is not stable. It follows that the principal $H'^{\mathbb{C}}/\ker(f)$-bundle obtained by extension of structure group via $H'^{\mathbb{C}} \to H'^{\mathbb{C}}/\ker(f)$ is also unstable. Thus, since E is obtained by extension of structure group via $f \colon H'^{\mathbb{C}}/\ker(f) \to H^{\mathbb{C}}$, we may assume that f is injective. The result now follows from [GO16, Proposition 3.13]; note that this is result about G-Higgs bundles but of course also applies to principal bundles, viewed as Higgs bundles with vanishing Higgs field. □

Remark 3.8. Note that the case of surjective f is equally valid for G-Higgs bundles, with essentially the same proof as that of [Ra75, Proposition 7.1]. Thus Proposition 3.7 in fact applies to G-Higgs bundles as well. We shall, however, not need this.

The following result shows the existence of G-Higgs bundles $(E, \varphi) \in \mathcal{F}_\lambda$, for $\lambda \neq 0$ as in the hypothesis of Corollary 3.5:

Proposition 3.9. *Let G be a non-abelian (real or complex) reductive connected Lie group. Then there exists a semistable Higgs bundle $(E, \varphi) \in \mathcal{M}(G)$ with E an unstable principal $H^{\mathbb{C}}$-bundle.*

Proof. If G is complex, consider the $\mathrm{SL}(2, \mathbb{C})$-Higgs bundle

$$(K^{1/2} \oplus K^{-1/2}, \varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}).$$

Clearly this is a stable $\mathrm{SL}(2, \mathbb{C})$-Higgs bundle and the underlying $\mathrm{SL}(2, \mathbb{C})$-bundle $K^{1/2} \oplus K^{-1/2}$ is unstable. Now we take the extended G-Higgs bundle via (3.2) which we denote by (E, φ). By Proposition 3.6 this is semistable and by Proposition 3.7 E is an unstable principal G-bundle.

For G real, we can use a variation of the same idea. Consider the basic $\mathrm{SL}(2, \mathbb{R})$-Higgs bundle

$$(K^{1/2} \oplus K^{-1/2}, Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \tag{3.6}$$

[2]The semisimple assumption on $H'^{\mathbb{C}}$ is a subtle point: for example, a line bundle L with $\deg(L) \neq 0$ is 0-unstable, however, there is no reduction to a proper parabolic of the structure group $\mathbb{C}^*$.

where 1 is the canonical section of $\mathrm{Hom}(K^{1/2}, K^{-1/2} \otimes K)$. Clearly this is a stable $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle.

Let (E, φ) be the G-Higgs bundle obtained from the basic $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle (3.6) via (3.5). Then, since the diagram

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\rho} & G \\ \downarrow & & \downarrow \\ \mathrm{SL}(2,\mathbb{C}) & \xrightarrow{\rho} & G^{\mathbb{C}} \supset H^{\mathbb{C}} \end{array}$$

commutes, we can use the argument of the previous paragraph to conclude that the $G^{\mathbb{C}}$-Higgs bundle $(\tilde{E}, \tilde{\varphi})$ obtained from (E, φ) by extension of structure group via $G \subset G^{\mathbb{C}}$ is a semistable $G^{\mathbb{C}}$-Higgs bundle, whose underlying principal $G^{\mathbb{C}}$-bundle $\tilde{E}$ is unstable. Finally note that $\tilde{E}$ is obtained from E by extension of structure group via the inclusion $H^{\mathbb{C}} \subset G^{\mathbb{C}}$. Hence the principal $H^{\mathbb{C}}$-bundle is also unstable (cf. Proposition 3.6). □

3.3. **Reducibility of the nilpotent cone.** Here we deduce reducibility of the nilpotent cone when G is a connected reductive complex Lie group. Thus, in this subsection $G = H^{\mathbb{C}}$.

Proposition 3.10. *Let G be a non-abelian connected reductive complex Lie group. Then the topological type of the extended G-Higgs bundle (E, φ) constructed in Proposition 3.9 is zero.*

Proof. The topological type of the basic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle:

$$(K^{1/2} \oplus K^{-1/2}, \varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$$

which we consider in the proof of Proposition 3.9 is zero and hence the topological type of the extended G-Higgs bundle is zero as well, by the induced homomorphism between the fundamental groups $i_* : \pi_1(\mathrm{SL}(2,\mathbb{C})) \to \pi_1(G)$ which is indeed trivial in this case. □

For the proof of the next result we shall need the notion of very stable G-bundles which we recall from [La88, BR94]: A principal G-bundle P is said to be *very stable* if $H^0(X, \mathrm{ad}P \otimes K)$ does not contain any non-zero nilpotent Higgs field.

Proposition 3.11. *Let G be a non-abelian connected reductive complex Lie group. Then, the nilpotent cone contains a component which does not belong to $\mathcal{N}_0(G)$.*

Proof. It follows from Proposition 3.9 and Proposition 3.10 that there exists a semistable G-Higgs bundle (E, φ) of trivial topological type for which E is unstable as a principal $H^{\mathbb{C}}$-bundle. This implies that there is some $\lambda \neq 0$ such that F_λ is non empty, see proof of Corollary 3.5. And on the other hand, using the existence of very stable G-bundles result, [BR94, Corollary 5.6], we can conclude that $\mathcal{F}_0$ is not contained in $\bar{D}_\lambda$ and hence the result follows. □

Theorem 3.12. *Let G be a non-abelian connected reductive complex Lie group. Then the nilpotent cone in the moduli space $\mathcal{M}_0(G)$ of G-Higgs bundles of trivial topological type is not irreducible.*

Proof. It is immediate from Proposition 3.11. □

Remark 3.13. The above result was shown in [BF11] in the semisimple complex case. Our result extends this to the complex reductive case. Since in the case of real reductive G we do not have existence result of very stable G-bundles we could not conclude reducibility of the nilpotent cone for this case.

3.4. **Non retracting and topological type.** By putting together our previous result here we prove that the moduli space of G-Higgs bundles does not deformation retract onto the moduli space of principal bundles. Since we want to study the obstructions to a deformation retraction from $\mathcal{M}(G)$ to $\mathcal{N}(H^{\mathbb{C}})$, we should consider separately each topological type, and here we consider trivial topological type. Thus, in order to apply Corollary 3.5, we should look for a semistable G-Higgs bundle (E, φ) in $\mathcal{M}_0(G)$ for which E is unstable as a principal $H^{\mathbb{C}}$-bundle. The following result shows that Proposition 3.9 gives a topologically trivial G-Higgs bundle with unstable underlying $H^{\mathbb{C}}$-bundle.

Proposition 3.14. *We have the following:*

(i) *Let G be a non-abelian connected simple real Lie group of Hermitian non-tube type. Then the topological type of the extended G-Higgs bundle (E, φ) constructed in Proposition 3.9 is zero.*

(ii) *Let G be a non-abelian connected reductive real Lie group of non-Hermitian type. Then there is a polystable G-Higgs bundle (E, φ) of trivial topological type such that E is an unstable $H^{\mathbb{C}}$-bundle.*

Proof. *Part* (i) follows from [GPR15, Proposition 7.1, Proposition 7.2]. To prove *Part* (ii), let $\tilde{G}$ be the universal cover of G and hence we have a surjective Lie group homomorphism

$$p : \tilde{G} \to G$$

such that $\ker(p)$ lies in the center of $\tilde{G}$. By Proposition 3.9 we obtain a polystable $\tilde{G}$-Higgs bundle $(\tilde{E}, \tilde{\varphi})$ with unstable $H^{\mathbb{C}}$-bundle and since $\tilde{G}$ is simply-connected the topological type of $\tilde{E}$ is trivial. Therefore, by using Proposition 3.7 and Proposition 3.6 the extended G-Higgs bundle via the covering map is the desired G-Higgs bundle. □

Remark. When G is a connected simple real Lie group of Hermitian tube type then the topological type of the extended G-Higgs bundle (E, φ) as in Proposition 3.9 is maximal, see [GPR15, Proposition 7.2]. Since we are studying the obstructions to a deformation retraction from the moduli space of polystable G-Higgs bundles of trivial topological type $\mathcal{M}_0(G)$ to $\mathcal{N}_0(H^{\mathbb{C}})$ we exclude this case in the above Proposition.

Finally putting our results together we obtain the following theorem. Note that the moduli spaces are generally singular.

Theorem 3.15. *Let G be a non-abelian (real or complex) connected reductive Lie group of non-Hermitian type or connected simple real Lie group of Hermitian non-tube type. Then the moduli space of semistable principal $H^{\mathbb{C}}$-bundles of trivial topological type $\mathcal{N}_0(H)$ is not a deformation retraction of the moduli space $\mathcal{M}_0(G)$ of semistable G-Higgs bundles of trivial topological type.*

Proof. Combine Corollary 3.5, Proposition 3.9, Proposition 3.10 and Proposition 3.14. □

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